Invariant Bilinear form

Casimir element

IF V IS A $\mathfrak{g}$-MODULE A SYMMETRIC BILINEAR FORM $(\mid) : V \times V \to \mathbb{C}$ IS INVARIANT IF

$$([x,y] \mid z) = (x \mid [y,z])$$

$$([x,y] \mid z) = -(y \mid [x,z])$$

IF $\mathfrak{g} = \mathrm{Lie}(G)$ AND

$$(gx \mid gy) = (x \mid y) \text{ THEN}$$

IT IS INVARIANT FOR THE INDUCED $\mathfrak{g}$ REP'N

$$0 = \frac{d}{dt}\left(e^{tX} v \mid e^{tY} w\right) = (X \cdot v \mid w) + (v \mid X \cdot w).$$

EXAMPLE: IF $\mathfrak{g}$ IS A F.D. SIMPLE LIE ALGEBRA THE KILLING FORM IS AN INVARIANT BILINEAR FORM ON $\mathfrak{g}$. $(x \mid y) = \mathrm{tr}(\mathrm{ad}\,x\, \mathrm{ad}\,y)$.

(SIMPLE $\Rightarrow$ NONDEGENERATE.)

THE CASIMIR OPERATOR.

THE CASE OF A FINITE-DIMENSIONAL SIMPLE LIE ALGEBRA.

IN THIS CASE THE CASIMIR $\Omega \in U(\mathfrak{g})$.

LET X_i BE A BASIS OF $\mathfrak{g}$

X^i DUAL BASIS.

ASSUMING $(\ ,\ )$ IS A ad-INVARIANT SYMMETRIC BILINEAR FORM ON $\mathfrak{g}$.

$$\Omega = \sum X_i X^i.$$

THEOREM: Ω IS INDEPENDENT OF BASIS AND Ω COMMUTES WITH $\mathfrak{g}$

(IN THIS CASE THAT IS THE SAME AS SAYING $\Omega \in Z(U(\mathfrak{g})))$.

OMIT VERIFICATION INDEPENDENT OF BASIS.

$$[z, X_i] = \sum_j c_{ij} X_j$$

$$[z, X^i] = \sum d_{ij} X^j$$

$$([z, x_i] \mid x^j) = c_{ij}$$

$$= (x_i \mid \underbrace{[z, x^j]}_{\sum d_{jh} x^h}) = -d_{ji}$$

So $\quad c_{ij} = -d_{ji}$

$$[z, x_i] = \sum_j c_{ij} x_j$$

$$[z, x^i] = -\sum c_{ji} x^j$$

$$[z, \Omega] = \sum_i [z, x_i] x^i + x_i [z, x^i]$$

$$= \sum_{i,j} c_{ij} x_j x^i + \sum x_i (-c_{ji}) x^j = 0.$$

THIS MEANS IF V IS IRREDUCIBLE, Ω ACTS BY A SCALAR.

ALSO TRUE FOR MORE GENERAL REP'S E.G. HIGHEST WEIGHT MODULES.

IF μ IS A PRIMITIVE WEIGHT IN $M(\lambda)$
Ω ACTS BY SAME SCALAR ON $M(\mu)$, $M(\lambda)$.
THIS CAN HELP FIND PRIMITIVE VECTORS
IF WE KNOW HOW THE SCALAR DEPENDS
ON λ.

$$0 \to M(\mu) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \to 0$$

sl_2 CASE.

LEMMA: IF $(\,,\,)$ IS INVARIANT
BILINEAR FORM ON $\mathfrak{g}$ AND

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}_\alpha \qquad \mathfrak{g}_0 = \mathfrak{h}$$

$$\mathfrak{g}_\alpha = \mathbb{C} x_\alpha$$

IF $x \in \mathfrak{g}_\alpha$ AND
IF $\alpha \in \Delta$.
$y \in \mathfrak{g}_\beta$, THEN $(x, y) = 0$.
UNLESS $\alpha = -\beta$.

IF $\alpha \neq -\beta$ FIND $H \in \mathfrak{h}$ WITH $\alpha(H) \neq -\beta(H)$

$$([H, x] \mid y) = -(x \mid [H, y])$$

$$\overset{\shortparallel}{\alpha(H)(x \mid y)} \qquad\qquad \overset{\shortparallel}{-\beta(H)(x \mid y)}$$

THEOREM: IF V IS A HW MODULE WITH WEIGHT λ, Ω ACTS AS A SCALAR ON V WITH VALUE

$$(\lambda \mid \lambda + 2\rho).$$

FIND BASIS H_i OF $\mathfrak{h}$

$X_\alpha \in \mathfrak{X}_\alpha$ DUAL BASIS.

$\alpha \in \Delta$ $(\alpha \neq 0)$ $H^i \in \mathfrak{h}$, $X^\alpha \in \mathfrak{X}_{-\alpha}$

SINCE $\alpha \neq 0$, $\alpha \neq -\alpha$ SO WE MAY

CHOOSE $X^\alpha = X_\alpha$ THEN

$$(X_\alpha | X_\beta) = \delta_{\alpha,-\beta} \qquad (H_i | H^j) = \delta_{ij}.$$

THEOREM: IF V IS A HIGHEST WEIGHT

MODULE WITH HIGHEST WEIGHT λ

AND SCALAR EIGENVALUE OF Ω IS $(\lambda | \lambda + 2\rho)$.

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

WE'LL SHOW $\Omega v_\lambda = (\lambda | \lambda + 2\rho) v_\lambda$ FOR HWV v_λ.

SINCE IT COMMUTES WITH $\mathfrak{g}$.

LET H_i BASIS OF $\mathfrak{h}$, H^i DUAL BASIS OF $\mathfrak{h}^*$.

$$\Omega = \sum_i H_i H^i + \sum_{\alpha \in \Delta^+} \underset{\sim\sim\sim\sim}{X_{-\alpha} X_\alpha} + \underline{X_\alpha X_{-\alpha}}$$

NOTICE $[X_\alpha, X_{-\alpha}] := H_\alpha \in \mathfrak{h}$.

I WANT TO WRITE

$$\Omega = \sum H_i H^i + \underset{\sim\sim\sim}{\sum_{\alpha \in \Delta^+} H_\alpha} + 2 \sum_{\alpha \in \Delta^+} X_{-\alpha} X_\alpha$$

THIS IS THE EXPRESSION THAT WILL GENERALIZE TO KM CASE.

NOTICE IF Ωv_λ THE LAST TERM PRODUCES ZERO BECAUSE $\alpha \in \Delta^+$ $X_\alpha v_\lambda = 0$ SINCE $X_\alpha \in \mathfrak{n}_+$,

WE HAVE TO PROVE $\left(\sum H_\alpha\right) v_\lambda$

$= (\lambda | 2\rho) v_\lambda$

AND $\sum H_i H^i v_\lambda = (x|\lambda) v_x$.

THERE IS AN ISOMORPHISM

$\nu : \mathfrak{g} \to \mathfrak{g}^*$ $\quad\quad \langle \mathfrak{h}, \mathfrak{g}^* \rangle$

$\langle x, \nu(y) \rangle = (x|y)$.

$\mathfrak{g} \ \mathfrak{h}$

$(\mathfrak{g}|\mathfrak{g}) \quad\quad (\mathfrak{g}^*|\mathfrak{g}^*) \quad\quad (\ |\ )$

INNER PRODUCE

$(\nu(\lambda)|\nu(\mu)) = (\lambda|\mu)$

ν IS ISO AND $(\ |\ )$ IS NONDEG

LEMMA: $\nu(H_\alpha) = \alpha$

NEED TO KNOW $H_\alpha = [X_\alpha, X_{-\alpha}]$

AND $(X_\alpha | X_{-\alpha}) = 1$ SO

$$= \langle \underset{\mathfrak{h}}{H}, \underset{\mathfrak{h}^*}{\gamma(H_\alpha)} \rangle = (H | H_\alpha)$$

$$= (H | [X_\alpha, X_{-\alpha}])$$

$$= ([H, X_\alpha] | X_{-\alpha}) \cdots$$

$$= \alpha(H) (X_\alpha | X_{-\alpha})$$

$$= \alpha(H).$$

$$= \langle H, \alpha \rangle.$$

SINCE THIS IS TRUE FOR ALL H,

$$\gamma(H_\alpha) = \alpha.$$

$$\Omega = \sum H_i H^i + \sum_{\alpha \in \Delta^+} H_\alpha + 2 \sum_{\alpha \in \Delta^+} X_{-\alpha} X_\alpha$$

$$= \sum H_i H^i + \gamma\left(\sum_{\alpha\in\Delta^+} \alpha\right) + 2\sum X_{-\alpha} X_\alpha$$

$$\underset{\text{STILL MYSTERIOUS.}}{\uparrow} \qquad \underset{\gamma(2\rho)}{\uparrow} \qquad \underset{\text{KILLS } v_\lambda.}{\uparrow}$$

LEMMA: IF $\lambda, \mu \in \mathfrak{h}^*$

$$\sum \lambda(H_i)\mu(H^i) = (\lambda | \mu).$$

PROOF:

$$H = \sum (H | H^i) H_i$$

$$(H | H') = \sum (H | H^i)(H' | H_i) \checkmark$$

$$(\lambda | \mu) = (\gamma^{-1}(\lambda) | \gamma^{-1}(\mu))$$

$$= \sum (\gamma^{-1}(\lambda) | H_i)(\gamma^{-1}(\mu) | H^i)$$

$$(\lambda|\mu) = \sum \lambda(H_i)\,\mu(H^i).$$

$$= \left(\sum H_i H^i + \gamma\Big(\sum_{\alpha\in\Delta^+}\alpha\Big) + 2\sum X_{-\alpha}X_{\alpha}\right) v_\lambda$$

$$H v_\lambda = \lambda(H)\, v_\lambda$$

$$\sum \lambda(H_i)\,\lambda(H^i)\, v_\lambda$$

$$+\ \gamma(2\rho)\, v_\lambda \quad + \ \text{ZERO}$$

$$\Big((\lambda|\lambda) + (\lambda|2\rho)\Big) v_\lambda$$

AS ADVERTISED.

START ON KM CASE.

DEFINE THE INNER PRODUCT ON $\mathfrak{h}$:

(ASSUME C.M. IS SYMMETRIC

NEED SYMMETRIZABLE $A = D \cdot B$.)

$$D = \begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_r \end{pmatrix}$$

THEOREM: THERE IS AN INVARIANT I.P. ON $\mathfrak{g}$. ON $\mathfrak{h}$:

$$(\alpha_i^\vee \mid \alpha_j^\vee) = \delta_{ij}$$

$$(\alpha_i \mid \alpha_j) = \delta_{ij}$$

($\alpha_i^\vee$ DON'T SPAN $\mathfrak{h}$.)

ASSUMING THIS WE CAN DEFINE CASIMIR ELT ON ANY MODULE IN CATEGORY $\mathcal{O}$. LET H_i, H^i BE DUAL BASES OF $\mathfrak{g}$. X_α^{j} BE BASES OF $\mathfrak{X}_\alpha$ ← 1-DIM'L FOR REAL ROOTS BUT NOT 1-DIM'L IN CENTER

$$\Omega = \sum H_i H^i + 2 \sum_{\substack{\alpha \in \Delta^+ \\ j}} X_{-\alpha}^{j} X_\alpha^{j} + 2\nu(\rho)$$

$\mathfrak{X}_\alpha$ $\mathfrak{X}_{-\alpha}$

X_α^{j} $X_{-\alpha}^{j}$